101 Linear Algebra Trivia.

Degree Course: Computer Science, University of Bologna.

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Determine the truth value of the following Linear Algebra propositions.

- If the statement is true, provide a brief proof (also by recalling known definitions or theorems).
- If the statement is false, provide a counterexample.

Nota: The falsity of a proposition can also be proven by citing a theorem or property that contradicts its statement.

The questions must be answered *concisely* (the term "concise" is to be understood as the intersection between brevity and rigor). **Avoid schematic arguments.** Also keep in mind that the questions presented here do not require laborious calculations but rather a good (sometimes very good) conceptual understanding of Linear Algebra, always staying within the covered syllabus. **Notations.** Notation is inferred from the context. In any case:

- Unless otherwise specified, uppercase Latin letters $A, B, C, \dots$ are used to indicate square matrices. If multiple matrices appear in an exercise, they are to be understood as being of the same order.
- The indices n, m indicate non-zero natural values.
- $\mathcal{M}_n(\mathbb{K})$ indicates the space of matrices of dimension n with coefficients in $\mathbb{K}$ (typically $\mathbb{K} = \mathbb{R}$).
- Vectors are sometimes written in bold, sometimes not. There is no specific reason for this.
- We work in $\mathbb{R}^n$. If you then want to have fun in other fields/rings, do so (with caution).

The following questions could be an *integrative* part of the oral exam (sometimes one, more than one, or none). Students are encouraged to reason through them on their own, or even in groups, without using low-quality tools. The document is not structured for those who study by heart or mechanically.

Legend. ☕: slightly challenging but formative question. ☕☕: as before, but more so.

QUESTIONS.

0. I can have 240 generating vectors for $\mathbb{R}^{120}$.
1. ☕ Let $A \in \mathcal{M}_n(\mathbb{R})$, then A and A^tA have the same kernel and therefore the same rank.
2. A linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable only if it has n distinct eigenvalues.
3. $f : \mathbb{R}^3 \to \mathbb{R}^2$ (linear map) is necessarily surjective.
4. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be three vectors in $\mathbb{R}^5$. Then $\dim(\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}) = 3$.
5. Every endomorphism of a vector space $\boldsymbol{V}$ is also injective.
6. ☕ Given two matrices A, B, we have $\mathrm{rk}(A+B) < \mathrm{rk}(A) + \mathrm{rk}(B)$.
7. Let $A \in \mathcal{M}_n(\mathbb{R})$. If λ_0 is an eigenvalue of A, then λ_0^n is an eigenvalue of A^n.
8. If $\det(A) = 0$ then A is not diagonalizable.
9. Let O_n be the zero matrix of order n. Is O_n diagonalizable?
10. Let $v = \left(\begin{smallmatrix} 8 \\ 40 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} 3{,}2 \\ 100 \end{smallmatrix}\right)$ be vectors in $\mathbb{R}^2$, and g a linear map such that $gv = w$. Is it true that v is an eigenvector of g?
11. A matrix $A \in \mathcal{M}_4(\mathbb{R})$ has eigenvalues $\{-3;\ 2;\ 3;\ 10\}$. What is the value of its determinant?
12. If λ_0 is an eigenvalue of A, then $-\lambda_0$ is an eigenvalue of A^t.

13. If 11 is an eigenvalue of A, then 11^{-1} is an eigenvalue of A^{-1}.
14. ☕ If A^{-1} is diagonalizable, then A is also diagonalizable.
15. If λ_0 is an eigenvalue of A, then $\lambda_0 + k$ is an eigenvalue of kA, for a non-zero real k.
16. Write, if possible, two linearly dependent vectors where only one of the two is a multiple of the other.
17. Let a be a linear map with associated matrix A, such that $A = A^t$. Then a is always diagonalizable.
18. Let λ_0, μ_0 be respectively an eigenvalue of A and one of B. Then $\lambda_0 + \mu_0$ is an eigenvalue of $A + B$.
19. It is impossible for a matrix to have the characteristic polynomial $p(x) = x^2 + 1$.
20. Let A be a matrix such that $A^2 = A$. Find the eigenvalues of A.
21. If 0 is an eigenvalue of A, then A is not invertible.
22. If 0 is an eigenvalue of A, then L_A, the linear map of which A is the associated matrix, is not surjective.
23. Let V be a vector space, and $\mathsf{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} = V$. Then $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ are linearly independent.
24. If $\det(A) = p$, then $\det(A^n) = p^n$.
25. Every matrix is a linear combination of rank one matrices.
26. The empty set is the basis of $\ker(O_n)$, where O_n is the zero matrix of order n.
27. Let $f : \mathbb{R}^8 \to \mathbb{R}^5$ be a linear map with $f(\boldsymbol{e}_k) = \boldsymbol{w}_k$ for $1 \le k \le 5$ and $f(\boldsymbol{e}_k) = \boldsymbol{0}$ for $6 \le k \le 8$, where $\boldsymbol{e}_k$ are the vectors of the canonical basis of $\mathbb{R}^8$ and $\boldsymbol{w}_k$ are non-zero vectors. Then it is possible that $\dim(\ker f) = 7$.
28. If a linear map f is such that $\dim(\ker f) = \dim(\operatorname{im} f)$, then $\ker(f) = \operatorname{im}(f)$.
29. For the linear map $f(x, y) = (x - y, -x + y)$, we have that $\ker(f) = \operatorname{im}(f)$.
30. Let U, W be two vector subspaces of $\mathbb{R}^4$. Is it possible to have $U \cap W = \emptyset$?
31. Given two matrices A, B of the same order, $\operatorname{rk}(AB) = \operatorname{rk}(BA)$.
32. Give an example of a linear map $f : \mathbb{R}^p \to \mathbb{R}^q$ such that $\ker(f) \cap \operatorname{im}(f) = \{\boldsymbol{0}\}$.
33. Let U, W be two vector subspaces of $\mathbb{R}^4$. Give, if possible, an example of U, W such that $\dim(U \cap W) = 1$.
34. Does the vector space $\mathbb{R}^1$ have a basis? Why? If so, exhibit one of your choice.
35. Let V, W be vector subspaces of $\mathbb{R}^n$. Then $V \cup W$ is a vector subspace of $\mathbb{R}^n$.
36. Let Θ be a map whose action on a generic vector $\boldsymbol{w}$ produces a rotation of $\frac{\pi}{2}$ radians. What are the eigenvectors of Θ?
37. Let W be a vector space. Is an orthogonal basis of W also orthonormal? Why? Is the converse true?
38. Let $W \subseteq \mathbb{R}^{127}$ be a vector subspace. Is it true that $\dim(W) + \dim(W^\perp) = 127$? Justify your answer.
39. A linear map $f : \mathbb{R}^6 \to \mathbb{R}^2$ cannot be surjective.
40. Let A be a matrix such that $A^2 = O_n$ (zero matrix of order n). Then $A^3 = O_n$.
41. Let f be a linear map, $f : \mathbb{R}^{14} \to \mathbb{R}^{14}$ definitely diagonalizable. Then there always exists an orthonormal basis of $\mathbb{R}^{14}$ made up of eigenvectors of f.
42. Consider exercise 36. What are the eigenvectors of Θ^4?
43. ☕☕ Now consider another map, $\Xi : \mathbb{R}^2 \to \mathbb{R}^2$, such that its action on a generic vector rotates it by $220°$. For which values of n is every vector an eigenvector of the map Ξ^n?

44. Let $m > n$. For every pair of rectangular matrices $G \in \mathcal{M}_{m,n}(\mathbb{R})$, $H \in \mathcal{M}_{n,m}(\mathbb{R})$, the product GH is invertible and its inverse is $H^{-1}G^{-1}$.

45. ☕ Let $f : V \to W$ be an isomorphism. Provided appropriate bases are chosen, f is represented by the identity matrix.

46. Let A be an invertible matrix. We have $(A^t)^{-1} = (A^{-1})^t$.

47. Let f be an endomorphism. Then f is also an isomorphism.

48. Let A, B be two matrices having the same characteristic polynomial $p(x) = x^3 - 5x^2 + 6x$. Then there exists an invertible G such that $B = G^{-1}AG$.

49. There exists a linear map $f : \mathbb{R}^9 \to \mathbb{R}^9$ such that $\ker(f) = \operatorname{im}(f)$.

50. Establish the existence, or non-existence, of a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose representative matrix has the characteristic polynomial $p(\lambda) = \lambda^2 - 3\lambda + 2$, and having eigenvectors $v_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$.

51. Given a linear map f, $f : U \to V$, and a linear map $g : W \to V$ then $g \circ f$ exists and we have $(g \circ f) : U \to W$.

52. There exists $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\ker(f) = \mathsf{span}\left\{\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)\right\}$ and $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \notin \operatorname{im}(f)$.

53. If f is a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f^2 = 0$, then $\ker(f) \subseteq \operatorname{im}(f)$.

54. Determine a linear map g whose associated matrix has the characteristic polynomial $p(x) = x^4 - x^2$.

55. Let $E = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Does there exist a linear map f, $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\ker(f) = E$ and $\left(\begin{smallmatrix}2\\5\\7\end{smallmatrix}\right) \in \operatorname{im}(f)$?

56. Let $v = \left(\begin{smallmatrix}\pi\\2\pi\end{smallmatrix}\right)$ be an eigenvector of f, with f being a linear map. Then $w = \left(\begin{smallmatrix}\pi^2\\4\pi^2\end{smallmatrix}\right)$ is an eigenvector of f^2.

57. If f is an injective linear map, then $\ker(f + \mathtt{id}) \bigoplus \ker(f - \mathtt{id}) = \mathbb{R}^3$.

58. Let f be an endomorphism of $\mathbb{R}^3$ such that $\operatorname{im}(f) \subset \ker(f)$. Then 0 is the only eigenvalue of f.

59. ☕ Let F be the matrix having as rows the vectors $v_1 = e_1, v_2 = -e_3, v_3 = e_2$. Does there exist a subspace $E \subset \mathbb{R}^3$ of dimension two, such that $Fv \in E, \forall v \in E$? If so, determine it explicitly.

60. Give, if it exists, a diagonalizable endomorphism of $\mathbb{R}^3$ having the characteristic polynomial $p(t) = (1 - t)^3$ and different from the identity.

61. If f is a diagonalizable endomorphism of $\mathbb{R}^n$, then $\ker(f) \bigoplus \operatorname{im}(f) = \mathbb{R}^n$.

62. If f is a diagonalizable endomorphism of $\mathbb{R}^n$, then $\ker(f) \perp \operatorname{im}(f)$.

63. Let U, V, W be vector subspaces. What is Grassmann's relation for $\dim(U + V + W)$?

64. Let f be an endomorphism of $\mathbb{R}^3$ such that $\ker(f + \mathtt{id}) \bigoplus \ker(f - 2\,\mathtt{id}) = \mathbb{R}^3$. Then f is diagonalizable.

65. Let U, V be subspaces of $\mathbb{R}^8$ with $\dim(U) = 5$. Then $\dim(U + V) \le 6$.

66. Given U, V vector subspaces of $\mathbb{R}^4$, exhibit a vector subspace V' such that $U \cup V'$ is still a vector subspace.

67. ☕ Let $U = \mathsf{span}\left\{\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)\right\}$ and $V = \{(x, y, z) \in \mathbb{R}^3 : x = 2y = 3z\}$. Let $f : U + V \to U \cap V$ be a linear map. Write F, the representative matrix of f.

68. In $\mathbb{R}^3$, establish, arguing in each case, if there exists a set that satisfies the following properties.
    (a) Two linearly independent vectors that span.
    (b) Three vectors that do not span.
    (c) Generators that are not a basis.
    (d) A set that contains a basis but does not span.
    (e) Two vectors of which only one is dependent on the other.
    (f) A set that does not contain a basis but spans.
69. State whether $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ span $\mathbb{Q}^3$.
70. Does there exist a unique linear map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(0,1) = (2,4)$ and $\phi(1,1) = (2,4)$?
71. Let $S : V \to W$ be a linear map between two vector spaces over $\mathbb{R}$, with $\dim(V) = n$. If $\dim(\ker(S)) \geq k$, what are the possible dimensions of $\text{im}(S)$?
72. ☕ Let $V = \mathbb{R}[x]$ be the vector space of polynomials with real coefficients. Verify that the map $\delta : V \to V$ defined by $\delta(p(x)) = p'(x)$ is linear for every $p(x)$. What are the eigenvalues and eigenvectors of V?
73. If $\dim(\ker(f)) > 0$ then f has no zero eigenvalues.
74. Let $V = \mathbb{R}^3$, $W = \mathbb{R}_2[x]$ and $T : V \to W$ be the linear map defined by $T(p,q,s) = p + qx + sx^2$. Find a basis for $S \subset \mathbb{R}^3$ such that $T(S)$ is the set of polynomials having a stationary point at $x = 1$.
75. There exists a relationship between the rank and determinant of a matrix.
76. Given U, V subspaces of $\mathbb{R}^n$ with $U \subset V$. Then $U^\perp \subset V^\perp$.
77. In $\mathbb{R}^3$ let $W = \text{span}\{(1,1,1)\}$. Describe "in words" what the vectors of $W^\perp$ are.
78. Let $A \in \mathcal{M}_{3\times 5}(\mathbb{R})$. Let U be the space generated by the rows of A. Describe what $U^\perp$ is.
79. ☕ In $\mathbb{R}^2$ let the scalar product $\langle u, w\rangle = u_1 w_1 - u_2 w_2$ be defined, let $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $W = \text{span}\{u\}$.
    Show that $u \in W$ and $u \in W^\perp$.
80. Let f be a linear map such that $\text{im}(f) \subset \ker(f)$. Then $f^2 = 0$.
81. Let f be a linear map such that $\ker(f) \subset \text{im}(f)$. Then $f^2 = f$.
82. Let $A \in \mathcal{M}_4(\mathbb{R})$. Let it be known that $\det(A) = 17$ and that the characteristic polynomial is $p(x) = x^4 + ax^3 + bx^2 + cx + d$. What is the value of the coefficient d?
83. If it exists, exhibit a matrix B such that $AB = \texttt{id}$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
84. Let $f \in \text{End}(\mathbb{R}^n)$ and let $v \in \mathbb{R}^n$, $u \in \text{im}(f)$, $w \in \ker(f)$. $f^2 = f$ if and only if $v = u + w$.
85. There exists a linear map $f : \mathbb{R}^7 \to \mathbb{R}^7$ that does not possess real eigenvalues.
86. There exists a linear endomorphism f that does not possess complex eigenvalues.
87. ☕ Let h be a linear endomorphism, and let it be known that $\ker(h) \not\subseteq \text{im}(h)$. Establish whether the following are true or false:
    - h is not invertible.
    - $\exists v \neq \mathbf{0} : h(v) = 0$.
    - h is diagonalizable.

88. ☕☕ Let g be a linear map, $g : \mathbb{R}^n \to \mathbb{R}^m$. Let $\boldsymbol{w}, \boldsymbol{v}$ be such that $g^t(\boldsymbol{w}) = 0$ and $\boldsymbol{v} \in \mathrm{im}(g)$. Then $\boldsymbol{w} \cdot \boldsymbol{v} = 0$.

89. Let U, V be subspaces of $\mathbb{R}^5$. Establish whether the following are true or false:

    - $\dim(U + V) > 4$.
    - $\dim(U) + \dim(V) = 3$ with $\dim(U + V) = 3$.
    - If $\{v_1, v_2, v_3, v_4, v_5\}$ is a basis of V, then it is also a basis of $V^{\perp}$.

90. A basis of $V^{\perp}$, with $V = \{(x, y, z, t) \in \mathbb{R}^4 : 3x - 6y + 3t = 0\}$, is $\{(1, -2, 0, 1)\}$.

91. ☕ There exists a linear endomorphism $f : V \to V$ such that $f^3 = f$ and $(f + \mathtt{id})^2 = 0$.

92. There exists a linear endomorphism $g : V \to V$ such that $g^4 = g$ and $g^2 = g + \mathtt{id}$.

93. If $\boldsymbol{w} = (1, 3, 5) \in \ker(f)$, with $f : V \to V$ being a linear endomorphism, and $f(9, 9, 9) = (5, 3, 1)$, then $(10, 12, 14) \in f^{-1}(5, 3, 1)$.

94. If $f : \mathbb{R}^m \to \mathbb{R}^m$ is a linear endomorphism such that $\sum_{k=0}^{m-1} \ker(f - k\,\mathtt{id}) = m$, then f is diagonalizable.

95. ☕ There exists a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{im}(f) = \mathsf{span}\{e_1, e_2\}$, $\ker(f) \subseteq \mathrm{im}(f)$ and such that $f^3 = 0$ but $f^2 \neq 0$.

96. Let $V = \mathbb{R}[x]_{\leq 3}$ be the vector space of polynomials of degree at most three, and $f : V \to V$ the linear map defined by $f(p(x)) = p''(x) + 5p'(x)$. Calculate $\det(f)$.

97. There exist $p, q \in \mathbb{N}$ with $p + q = 10$, such that $f : \mathbb{R}^{p+4} \to \mathbb{R}^{q-8}$ is surjective.

98. ☕ Let $g : \mathbb{R}^8 \to \mathbb{R}^0$ be a linear map. Find $\mathrm{im}(g)$. What is the matrix associated with g?

99. ☕ There exists an invertible $f : \mathbb{R}^0 \to \mathbb{R}^n$.

100. Let f, g be linear maps such that $\dim(\ker(f)) = 3 - \dim(\mathrm{im}(g))$, and $\mathrm{im}(f) \subseteq \ker(g)$. Then g is surjective or f is injective. Under the same hypotheses, is it true or false that f and g must possess at least one zero eigenvalue?