29 additional Linear Algebra Trivia for enthusiasts.

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The following document arises from a mere statistical matter: it is conjectured[1] that in a class of N people, about $\sqrt{N}$ become strongly passionate about the course to the point of wanting to study the subject in more depth. It is an average and, like any statistical concept, should be taken with a grain of salt and a bit of irony. Enthusiasts who want to have fun (possibly at night) can try their hand at these additional little games.
(The worlds of Linear Algebra are infinitely more extensive than what can be read here)

Please note that what appears in this document is NOT part of the exam. It is an extra, independent of the course.

Some notations or terminologies:

- $\mathbb{K}$ indicates a field, generally $\mathbb{R}$ or $\mathbb{C}$ unless otherwise specified.
- $\hom(\bullet,\bullet)$ means "homomorphisms" or sometimes "set of homomorphisms" (with relative specification of starting and arriving vector space)
- $\mathrm{End}(\bullet)$ is the set of endomorphisms of a given vector space $\bullet$ (the field is implicitly defined).
- The identity matrix can be denoted in the following ways: $\mathtt{id}$, I, $\mathbb{1}$.

Questions.

0. Construct all invertible matrices $H \in \mathcal{M}_2(\mathbb{R})$ such that $HH^T = \mathtt{id}_2$.
1. Let V be the vector space of polynomials with real coefficients of degree at most 5. Let W be the vector subspace of V consisting of polynomials $p(x)$ such that $p(x) = p(-x)$. Calculate $\dim(W)$. Find a subspace T of V such that $W \bigoplus T = V$.
2. Let $B \in \mathcal{M}_2(\mathbb{Q})$ be the matrix
$$B = \begin{pmatrix} n^2 & -1 \\ p & n-p \end{pmatrix}, \qquad n, p \in \mathbb{Q}$$
Show that $B^2 = \mathbf{0}$ if and only if $n = p = 0$.
3. Let $f : \mathbb{R}^{350} \to \mathbb{R}^{250}$ be a linear map, and $V \subseteq \mathbb{R}^{350}$ a subspace such that $\dim(V) = 300$ and $\dim(V \cap \ker(f)) = 50$. Determine $\dim(\mathrm{im}(f))$ and establish whether f is surjective or not.
4. Let V be a vector space over $\mathbb{Q}$ and let $f \in \mathrm{End}(V)$ be an endomorphism with characteristic polynomial $p_f(x) = x^3 - 3x^2 + 2x$.
   (a) Calculate the characteristic polynomial of $g = f + (f + 2I)^{-1}$, and determine the algebraic and geometric multiplicity of each eigenvalue.
   (b) Repeat in $V = \mathbb{Z}/3\mathbb{Z}$
5. Give an example of a matrix $A \in \mathcal{M}_n(\mathbb{R})$ such that A^2 is diagonalizable and A^3 is not.
6. Let $A \in \mathcal{M}_n(\mathbb{K})$. Show that $\lambda \in \mathbb{K}$ is an eigenvalue of A if and only if it is an eigenvalue of A^T.
7. Let $A \in \mathcal{M}_n(\mathbb{R})$. Show that if A is a symmetric nilpotent matrix, then $A = \mathbf{0}$.
8. Let $\mathbb{K} = \mathbb{Z}/11\mathbb{Z}$ and let $f : \mathbb{K}^8 \to \mathbb{K}^6$ be a linear map such that $\ker(f)$ has 121 elements. Then f is surjective.

[1] Being a conjecture, there is no actual proof other than the observational one.

9. Let f be an endomorphism of a vector space V. Show that
   (a) $\ker(f) \subseteq \ker(f^2)$
   (b) $f\big(\ker(f^2)\big) \subseteq \ker(f)$

   Then consider the restriction of f to the kernel of f^2 and show that
   (c) $\dim\big(\ker(f^2)\big) \leq 2\dim\big(\ker(f)\big)$
10. True or false? Let $W \subset \mathbb{K}^5$ be a subspace of dimension 2. Then we can write W as the intersection of two subspaces of dimension 4.
11. Determine eigenvalues and eigenspaces of $f : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$, with $f(A) = A - 3A^T$.
12. Let W_1, W_2, W_3 be vector subspaces of a vector space V such that $V = W_1 \bigoplus W_2 = W_1 \bigoplus W_3$. Then $W_2 = W_3$.
13. Let $C \in \mathcal{M}_n(\mathbb{R})$, and consider the set
$$\langle C\rangle = \big\{CM \mid M \in \mathcal{M}_n(\mathbb{R})\big\}$$
obtained by right-multiplying C by every possible square matrix of order n. Show that $\langle C\rangle$ is a vector subspace of $\mathcal{M}_n(\mathbb{R})$.
14. There are infinitely many vector subspaces in $\mathbb{Q}^2$.
15. Establish the existence of a linear map $f : V \to V$ such that f is injective, $f^2 = 2f - \mathtt{id}$ and $f^{-1} = \alpha f + \beta\,\mathtt{id}$ for some real α, β. If such a map exists, construct an example in $V = \mathbb{R}^2$.
16. Give, if possible, two vector subspaces U, V such that $U + V = \mathbb{R}^4$ where the sum is not direct.
17. Establish the values of the parameter $a \in \mathbb{R}$ for which $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(x, y, z) = (ax + y + 2z, 2ax - z, -2y + az)$ is an automorphism.
18. Construct a map $f : V \to V$ with $V = \mathbb{R}^5$ such that $\dim(\ker(f^2 - 4f)) = 1$.
19. Find the eigenvalues and eigenvectors of the derivative operator $D := \frac{\mathrm{d}}{\mathrm{d}x}$, with $D : V \to V$, where $V = \mathcal{C}^\infty(\mathbb{R})$.
20. Let $W_1 = \big\{(t, t, t, -t), t \in \mathbb{Q}\big\}$ and $W_2 = \big\{(a + b + c, b - c, a - b, a - c) \mid a, b, c \in \mathbb{Q}\big\}$. Write the Cartesian equation of $W_1 \cap W_2$.
21. Does there exist a nilpotent and non-surjective automorphism on a vector space of even dimension? Does there exist a nilpotent automorphism on a vector space of even dimension?
22. Calculate $\dim(V)$ for $V = \big\{f \in \hom(\mathbb{R}^4, \mathbb{R}^4) \mid f(e_1) = 2f(e_2), f(e_4) \in \mathsf{span}\big\{(1, 1, 1, 1)\big\}\big\}$
23. Show that the derivation $\frac{\mathrm{d}}{\mathrm{d}x} : \mathbb{K}[x]_{\leq 3} \to \mathbb{K}[x]_{\leq 3}$ is nilpotent.
24. Let $\mathbb{K}$ be a field and let $V = \mathbb{K}[x]_{\leq 4}$ be the vector space of polynomials with coefficients in $\mathbb{K}$ in one variable and of degree at most 4. Let f be the endomorphism defined by $f(a) = a'$, for every $a \in V$ (where a' denotes the usual derivative), and let $g = f + f^2$.
    Find the eigenvalues of f and g.
25. Establish whether the following matrix is diagonalizable in $\mathbb{R}$. Use only the theorems of Mathematical Analysis during the study of the roots of the characteristic polynomial. Do not use any calculator (it is possible, however, to make convenient numerical approximations, where possible and without exaggerating).
$$P = \begin{pmatrix} 1 & 0 & 4 \\ -2 & -1 & 8 \\ 0 & 6 & 6 \end{pmatrix}$$

26. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$f(x_1, x_2, x_3, x_4) = (ax_1 + x_3, bx_2 + x_4, x_1 - x_3 + ax_4, ax_1 - bx_2 + x_3), \qquad a, b \in \mathbb{R}$$

Establish for which values of a, b, f is

(a) An isomorphism.

(b) A non-injective epimorphism.

(c) An automorphism.

(d) A diagonalizable endomorphism.

27. Prove that if $f : V \to W$ and $g : W \to Z$, then $g \circ f = 0 \iff \mathrm{im}(f) \subset \ker(g)$.

28. Let f, g, h be linear maps on a vector space V of dimension N. Let it be known that

(1) $\dim(\ker(f)) + \dim(\ker(g)) = \dim(\mathrm{im}(h)) + 4$

(2) $\mathrm{im}(f) \subseteq \mathrm{im}(h)$

(3) $\ker(g) \subset \ker(f)$

(4) $\dim(\mathrm{im}(h)) - \dim(\mathrm{im}(g)) = \dim(\mathrm{im}(f)) - 6$

Establish whether it is possible that $\ker(f), \ker(g), \ker(h)$ have the same dimensions. Establish whether it is possible that $\mathrm{im}(f), \mathrm{im}(g), \mathrm{im}(h)$ have the same dimensions. Regardless of what was found in the two previous points, establish the minimum value of $N = \dim(V)$ for which such a triple of linear maps can exist.