# Problems and Exercises in Discrete Mathematics.

Something to have fun with.

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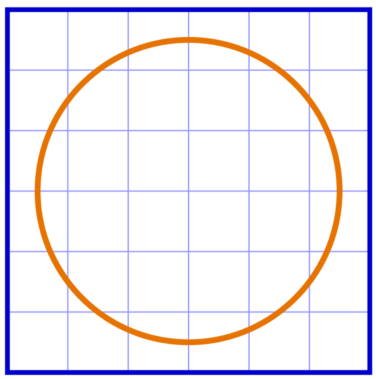
**M.0.** Suppose there are $2n$ people arranged around a circle; the first $n$ are "good people", and the last $n$ are "bad people". Show that there always exists an integer $m$ depending on $n$ such that if we go around the circle killing one person every $m$, all the bad men are the first to be eliminated (example: when $n = 3$ we can take $m = 5$; when $n = 4$ we can take $m = 30$).

**M.1.** Calculate the sum $\sum_{k\geq 1} \frac{(-1)^k k}{4k^2-1}$

**M.2.** "Cribbage" players have long known that $15 = 7+8 = 4+5+6 = 1+2+3+4+5$. Find the number of ways to represent 1050 as a sum of consecutive positive integers (the trivial representation 1050 alone counts as one way; thus there are four ways to represent 15 as a sum of consecutive positive integers. Incidentally, knowledge of the rules of "cribbage" is irrelevant to the problem.)

**M.3.** A circle, with a diameter of $2n-1$ units, has been drawn symmetrically on a $2n \times 2n$ chessboard shown here for $n = 3$:



(a) How many squares of the chessboard contain a piece of the circle?

(b) Find a function $f(k)$ such that exactly $\sum_{k=1}^{n-1} f(k)$ squares of the chessboard lie entirely inside the circle.

**M.4.** Simplify the expression $\lfloor (n+1)^2 n! e \rfloor \mod n$.

**M.5.** How many numbers of the form $2^m$, for $0 \leq m \leq M$, have 1 as their first digit in decimal notation?

**M.6.** What is the smallest positive integer that has exactly $k$ divisors, for $1 \leq k \leq 6$?

**M.7.** Show that $\frac{3^{77}-1}{2}$ is odd.

**M.8.** Determine the value of $1000! \mod 10^{250}$ through a manual calculation.

**M.9.** Show that if $2^n + 1$ is prime then $n$ is a power of 2.

**M.10.** Which binary number corresponds to Euler's number $e$, in the correspondence between binary representation and Stern-Brocot representation? (Express your answer as an infinite sum, it is not necessary to calculate it in a closed form).

**M.11.** Let $n$ be a given positive integer; for what values of $k$ is the binomial coefficient $\binom{n}{k}$ maximum? Prove the given answer.

**M.12.** Show that $-2(\ln(1-z)+z)/z^2$ is a hypergeometric function.

**M.13.** What is the value of $\sum_k \binom{n}{k}^3 (-1)^k$?

**M.14.** Find a closed form for $\sum_{k\le n}\binom{2k}{k}4^{-k}$

**M.15.** Find a closed form for

$$\sum_{k\ge1}\binom{n}{\lfloor\log_m(k)\rfloor}$$

when $m, n$ are positive integers.

**M.16.** Use Stirling's approximation to estimate $\binom{m+n}{n}$ for large $m, n$. Observe what happens when $m = n$.

**M.17.** Is it possible to build a stack of $n$ bricks in such a way that the top brick lies completely outside the bottom brick, while a person weighing as much as 100 bricks can balance in the middle of the top brick without causing the stack to fall?

**M.18.** Show that $\frac{\mathrm{d}}{\mathrm{d}z}(z!)$ is related to generalized harmonic numbers.

**M.19.** *Research problem.* Prove the irrationality of $\gamma$ and $e^\gamma$, where $\gamma$ is the Euler-Mascheroni constant.

**M.20.** How many ways are there to place the numbers from 1 to $2n$ in a $2\times n$ matrix so that the elements of the rows and columns are in increasing order from left to right and from top to bottom? For example, a solution when $n = 5$ is the following

$$\begin{pmatrix}1 & 2 & 4 & 5 & 8\\ 3 & 6 & 7 & 9 & 10\end{pmatrix}$$

**M.21.** What is the probability that the top card and the bottom card in a randomly shuffled deck are both aces? (Consider a standard 52-card deck).

**M.22.** Construct a random variable such that it has finite mean and infinite variance.

**M.23.** Which grows faster between $n^{\ln(\ln(\ln(n)))}$ and $(\ln(n))!$?

**M.24.** Calculate $\left(n+2+O(n^{-1})\right)^n$ with relative error $O(n^{-1})$.

**M.25.** True or false? $\cos\left(O(x)\right) = 1 + O(x^2)$ for all real $x$.

**M.26.** True or false? $O(x+y)^2 = O(x^2) + O(y^2)$.

**M.27.** Calculate the "Fibonacci factorial" $\prod_{k=1}^{n} F_k$ with a relative error of $O(n^{-1})$ or better. Your solution might contain a constant that you do not know in closed form.

**M.28.** True or false?

$$\int_n^{+\infty} O(x^{-2})\,\mathrm{d}x = O(n^{-1}), \qquad n\to+\infty$$

**M.29.** Prove that

$$B_m(\{x\}) = -2\frac{m!}{(2\pi)^m}\sum_{k\ge1}\frac{\cos\left(2\pi kx-\frac{1}{2}\pi m\right)}{k^m}, \qquad m\ge2$$

using the calculus of residues, integrating

$$\frac{1}{2\pi i}\oint\frac{2\pi i e^{2\pi i z\theta}}{e^{2\pi i z - 1}}\frac{\mathrm{d}z}{z^m}$$

on the contour of the square $z = x + iy$, where $\max\left(|x|,|y|\right) = M + \frac{1}{2}$, and letting $M$ tend to infinity.
( $B_m(\{x\})$ denotes the Bernoulli polynomials, and $\{x\}$ denotes the fractional part of $x$ ).

**M.30.** *The pigeons' election.* The pigeons of the United States of Piggeon are holding an election to determine who will be the Grand Pigeon. Elections in this state are quite peculiar due to a long and historical tradition called the "Electoral University".
Here are the rules:

1. The United States of Piggeon counts 50 states, and each pigeon belongs to one of them (obviously). There is a total of $2k^2 - 1$ pigeons in state $k$, for $1 \le k \le 50$.
2. There are two candidates for the title of Grand Pigeon: Pigeon and Pigeonacci.
3. Every pigeon votes for one of the two aforementioned candidates.
4. The candidate who receives the most votes in the $k$-th state gets $k$ votes at the Electoral University.
5. The candidate who gets the highest number of votes at the Electoral University becomes the Grand Pigeon.

That being said:

(a) Determine the *minimum* total number of votes necessary for any candidate to become the Grand Pigeon. Clarification: that is, find a number $p$ such that the Grand Pigeon always obtains at least $p$ votes from the voting pigeons, and such that a total of $p-1$ votes is never sufficient to win.

(b) Generalize the problem to the case of $4n+2$ states instead of 50. Provide a closed form expressing the minimum total number of votes, as discussed in the previous point, as a function of $n$.

**M.31.** Give a proof for the following identity:

$$\sum_{n\ge1} \frac{\cos(2\pi n x)}{n^2} = \pi^2\left(x^2 - x + \frac{1}{6}\right), \qquad 0 \le x \le 1$$

using only the “mathematics of Euler’s era” (18th century).