Problems and Exercises of Dynamical Systems and ODEs.
Degree Course in Physics + Mathematics.

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Some of the following problems and exercises are assigned periodically. It is formative and important to try to solve them all to be ready for the exam, although solving them all is neither a sufficient nor a necessary condition for passing the exam (but it certainly provides a great help).

P.0. On Earth at latitude ϕ, a bowl is launched on a smooth horizontal plane with velocity v_0 in the direction of the meridian. What is the acting force and what is the deviation from the meridian after a path of length ℓ?

P.1. A rocket is launched vertically from the Earth's surface with a velocity equal to the escape velocity. Neglecting the variation of the rocket's mass, find the equation of motion.

P.2. Find the central force necessary for a particle to describe a spiral of equation $r = e^{-\theta}$.

P.3. Given an attractive central field $f(r)$, determine the condition for a circular orbit to be stable.
Prove that if $f(r) = -ke^{-r^2}/r^2$, stability occurs only for $r^2 < 1/2$.

P.4. A point mass moves in a central force field along a circumference, on a point of which the center of the field itself is located.
Write the expression of the force and the potential, and determine the law $\phi = \phi(t)$ for $\phi \gg 1$.

P.5. At latitude ϕ, calculate the inclination of the plumb line with respect to the vertical.

P.6. A point mass moves on a cylindrical helix of radius R and pitch h, subject to its weight and a resisting force $\boldsymbol{F} = -2\beta \boldsymbol{v}$.
Determine the terminal velocity and the equation of motion if $\boldsymbol{r}(0) = R\hat{\imath}, \boldsymbol{v}(0) = 0$.

P.7. Determine the equation of motion for $t > 0$ of a point mass of unit mass, moving along the $\hat{x}$ axis under the action of the force $F = -\frac{V}{(1+t)} + 3(1+t)$ if the initial conditions are $v(0) = v_0, x(0) = 0$.

P.8. Find the trajectory for a heavy body using Maupertuis' principle.

P.9. A point mass is subject to the central force field with potential $V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$ with $\alpha > 0, \beta > 0$;

(a) determine the orbits;

(b) calculate the differential cross-section.

P.10. Determine the central force law that gives rise to the orbit $r(\varphi) = R(1-\cos(\varphi))$.

P.11. Consider the two Cauchy problems

$$A \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} + x\,\mathrm{tg}(t) = 0 \\ x(0) = 1 \end{cases}$$

$$B \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{1+xt} = 0 \\ x(0) = 0 \end{cases}$$

1. Indicate for which of the two it is possible to determine a closed-form solution.
2. Determine the solution referred to in point 1).
3. Solve both problems iteratively, stopping at the second iteration, using Picard's method, and compare with the exact solution referred to in point 1).

P.12. Given the central field

$$V(r) = \beta r^p$$

with $\beta, p \in \mathbb{R}$, $p \neq 0$, determine the conditions on the parameters β, p so that stable circular orbits can exist.

P.13. Study the orbits close to the circular orbit in the central field $V(r) = \alpha r^2$ with $\alpha > 0$.

P.14. Given a linear system of ordinary differential equations

$$\dot{x} = A(t)x$$

prove that the Wronskian W at time t is given by

$$W(t) = W(0)\exp\left[\int_0^t \operatorname{tr}\{A(u)\}\,\mathrm{d}u\right]$$

where $\operatorname{tr}\{A\}$ is the trace of the coefficient matrix. Deduce (proving the reason) that for the harmonic oscillator the Wronskian is constant.

P.15. Consider the one-dimensional motion of a point in the force field with potential

$$V(x) = -\frac{1}{2}\big(\sinh(x)\big)^4$$

(a) Determine the motion for initial conditions $x_0 > 0$, $v_0 < 0$ such that the total energy is zero.

(b) For the potential $V(x) = -\frac{1}{2}x^4$ that approximates the previous one in a neighborhood of the origin, determine the zero-energy solution, the stable and unstable manifolds of the critical point (phase space trajectories passing through the critical point), and qualitatively sketch the behavior of the phase space trajectories.

P.16. Consider the force field

$$\boldsymbol{F} = \frac{x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^2}\,\hat{\imath} + \frac{2xy}{(x^2 + y^2 + z^2)^2}\,\hat{\jmath} + \frac{2xz}{(x^2 + y^2 + z^2)^2}\,\hat{k}$$

1. Prove that it is a conservative field.
2. Find the components F_r, F_θ, F_ϕ of $\boldsymbol{F}$ in spherical coordinates.
3. Determine a potential for $\boldsymbol{F}$.

P.17. Solve the Cauchy problem consisting of the differential equation with constant coefficients

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = a(1 + bt)\cos(\Omega t)$$

accompanied by the initial conditions

$$x(0) = x_0$$

$$\dot{x}(0) = 0$$

assuming $0 < \beta < \omega$.

P.18. Given the plane curve with parametric equations in polar coordinates

$$r(t) = r_0 e^{\beta t}$$

$$\varphi(t) = \omega t$$

determine the curvilinear abscissa, the tangent unit vector, the normal unit vector, the binormal unit vector, and the radius of curvature.

P.19. Determine the coefficients of the Fourier expansion of the periodic function

$$f(t) = t - n, \qquad n \le t < n+1$$

P.20. Consider the one-dimensional motion of a point mass of unit mass subject to the potential

$$V(x) = \frac{|x|^\alpha}{\alpha}$$

with $\alpha > 1$. Letting E be the energy and T the period of motion, explicitly determine the function $T(E)$.

P.21. Identify and classify the equilibrium points of the following system of differential equations

$$\begin{cases} \dot{x} = x(-a+by), & x, b > 0 \\ \dot{y} = y(c-bx), & c > 0 \end{cases}$$

by calculating the linearized solutions and approximately sketching the phase trajectories.

P.22. Consider a one-dimensional motion governed by the force $F(x)$ defined by

$$F(x) = \begin{cases} -\frac{1}{2}\frac{\mathrm{sign}\,(x)}{\sqrt{|x|-1}}, & |x| > 1 \\ 0, & |x| < 1 \end{cases}$$

where $\mathrm{sign}\,(x)$ is the sign function.

(a) Study the equilibrium configurations.

(b) Sketch the phase trajectories.

(c) Calculate the period of motion $T(E)$ as a function of energy.

P.23. Solve the system of differential equations

$$\dot{x}_k = \sum_{i=1}^{k} x_i, \qquad k = 1, 2, \ldots, n$$

for assigned initial conditions $x_k(0)$.

P.24. Determine a central field $V(r)$ in which the azimuthal angle φ varies in time according to the law

$$\varphi(t) = \mathrm{arctg}(\omega t)$$

where ω is a positive constant. Interpret the result.

P.25. Starting from Maupertuis' principle and using Euler angles, derive the equations of the trajectory of motion for a gyroscopic rigid body placed in the gravity field.

P.26. Find the equation of the loxodrome on the sphere, i.e., the equation of the curve that, on the spherical surface, has its tangent vector forming a fixed angle α with the geodesics, i.e., with the meridians. Also calculate the length of this curve.

P.27. A point mass moves in a vertical plane, under the action of its weight, along a curve of class $\mathcal{C}^1$ whose Cartesian equation is

$$y(x) = \begin{cases} \log(a+bx^2) - \log(a), & |x| \le \sqrt{\frac{1}{|b|}} \\ \alpha x^2 + \beta|x|, & |x| \ge \sqrt{\frac{1}{|b|}} \end{cases}$$

where $a, b \in \mathbb{R}$ and $\alpha > 1$ is a fixed parameter.
Study the motion and the phase portrait as a function of the parameter b and calculate the period of motion for small oscillations around $x = 0$.

P.28. Determine the geodesics on the manifold with metric $\mathrm{d}s^2 = \sqrt{E-y}[\,\mathrm{d}x^2 + \,\mathrm{d}y^2]$.

P.29. Calculate the kinetic energy of a homogeneous equilateral triangle of side L and mass M rotating with constant angular velocity Ω around an axis passing through one of its sides.

P.30. Write the Hamilton function and find the constants of motion for an isotropic two-dimensional oscillator, that is, a point mass m subject to the force $\boldsymbol{F} = -k(x\hat{\imath} + u\hat{\jmath})$.

P.31. For the two-body problem, express in Hamiltonian formalism the transformation to relative and center-of-mass coordinates, and to the corresponding conjugate momenta, explicitly writing the canonical transformation and its generating function.

P.32. If $\mathcal{F}(X, p, t)$ is a constant of motion and the Hamiltonian H is also a constant of motion, prove that $\frac{\partial \mathcal{F}}{\partial t}$ is constant and equals $[H, \mathcal{F}]$.
In the case of a uniform force field $F_x = -mg$, prove the previous result if $\mathcal{F} = x - \frac{p}{m}t + \frac{1}{2}gt^2$.

P.33. A particle moves under the action of the force $F = k\frac{e^{-t/\tau}}{x^2}$ where k, τ are positive constants.
Calculate the Lagrangian and Hamiltonian of the system and the total energy, and discuss the conservation of energy.

P.34. A thin lamina of mass M is located in the (x, y) plane.

(a) Prove that the inertia tensor is of the form

$$I = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

(b) Determine the principal moments of inertia and the rotation angle that transitions from the x and y axes to the principal axes of inertia.

(c) Express A, B, C in the case of a lamina bounded by a closed curve with equation, in polar coordinates, $\rho = \rho(\varphi)$ and in the case of uniform density.

(d) Explicitly calculate A, B, C if $\rho = \sin(2\varphi)$, $0 \le \varphi \le \phi/2$.

P.35. Show that for any scalar field $\phi(x, y, z)$ of class $\mathcal{C}^2$, $\nabla \times (\phi \nabla \phi) = 0$ holds; at what point in the proof is it necessary to require that $\phi \in \mathcal{C}^2$?

P.36. Write the Lagrangian of a torus, made of homogeneous material, with mass M and radii r, R, rolling without slipping on a horizontal plane, expressing it explicitly as a function of the problem's data.

P.37. Given the Hamiltonian $H(p, q) = \frac{1}{8\sqrt{2}}\left(7p^2 - 2\sqrt{\alpha}pq + 5q^2\right)$ determine:

1. the set of values of the parameter $\alpha > 0$ for which the origin of the phase plane is a stable equilibrium point for the dynamics associated with H;
2. the frequency of motion for $\alpha = 3$, knowing that this value belongs to the set referred to in point 1);
3. for the same value of α, the canonical transformation that takes H into $H(j)$, where j is the action variable.

P.38. Given two pendulums of lengths l_i and masses m_i $(i = 1, 2)$ whose suspension points are at a distance L and located at the same height. The two pendulums interact with a torque $\boldsymbol{N} = -\alpha\hat{\boldsymbol{n}}(\phi_2 - \phi_1)$, where ϕ_1, ϕ_2 are the angular elongations of the two pendulums with respect to the vertical, $\hat{\boldsymbol{n}}$ is the normal unit vector to the plane of oscillation and α is a constant. Study how the characteristic oscillation frequencies depend on the value of α, and explicitly evaluate their limits for $\alpha \to 0$ and $\alpha \to +\infty$ indicating which mechanical systems these limiting cases correspond to.

P.39. Given the Hamiltonian

$$H = j_1 + \omega j_2 + \lambda(j_1 + \sin(\phi_1 - \phi_2))$$

calculate the first orders of perturbation theory and compare the result with the direct integration of Hamilton's equations.

P.40. Consider two point masses of equal mass, moving in a vertical plane under the action of gravity and connected by a rigid rod of negligible mass and length l. One of the two points is also constrained to move along a curve $y = f(x)$.

(a) Write the Lagrangian.

(b) Set the conditions for the existence of a stable equilibrium configuration.

(c) State which of the following curves satisfy the conditions given in the previous point b)

$$y = |a|^3 x^3$$

$$y = a^2 x^2$$

$$y = -|a|\sqrt{1 - \frac{x^2}{b^2}}$$

$$y = |x|$$

$$y = a^4 x^4$$

(d) Further select the curves for which the motion in the neighborhood of the stable equilibrium position can be described with the theory of small oscillations and write the corresponding approximated Lagrangian.

P.41. Determine the values of the angular momentum l for which finite orbits are possible in the following central potential

$$U(r) = -\frac{\alpha e^{-kr}}{r}$$

P.42. Consider the Hamiltonian

$$H = p^2 + x^{2n}$$

For a given energy $H = E$, determine:

(a) the period of motion;

(b) the action and the Hamiltonian as a function of the action.

(Denote by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t$$

the Euler Beta function.)

P.43. Consider a point mass subject to an elastic force and a nonlinear friction force that lead to the following equation of motion:

$$\ddot{x} + \dot{x}(ax^2 + \dot{x}^2 - 1) + ax = 0$$

Show that this equation has a limit cycle, specifying its geometric nature and illustrating its origin from a mechanical point of view.

P.44. Solve the problem of motion described by the Hamiltonian

$$H = \frac{p^2}{2} + gx$$

using the Hamilton-Jacobi equation.

P.45. Determine the intrinsic triad and the radius of curvature on the intersection curve of the following surfaces in $\mathbb{R}^3$:

$$\begin{cases} z = \log(x^2 + y^2), \quad x^2 + y^2 < \infty \\ y = x\,\mathrm{tg}(z) \end{cases}$$

P.46. A body moving in a central field describes an orbit whose equation, in polar coordinates, is

$$r(\phi) = \frac{R}{(\cos(\phi))^{2k} + (\sin(\phi))^{2k}}$$

with integer k and constant R.
Determine the potential of the field for $k = 1$ and for $k = 2$.

P.47. Given the Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{x^3}{3}$$

(a) determine the equilibrium positions and their stability;

(b) determine the equation of the separatrix;

(c) sketch (approximately) the phase portrait of the system;

(d) calculate, optionally, to the first significant order the variation of frequency with amplitude for orbits close to the stable equilibrium position.

P.48. With a suitable choice of the representative phase space, parameterized by the "Deprit variables", the Hamiltonian of a free rigid body with a fixed point can be reduced to the following:

$$H = \frac{L^2}{2I_3} + \frac{1}{2}\left(\frac{\sin^2(l)}{I_1} + \frac{\cos^2(l)}{I_2}\right)(G^2 - L^2)$$

where I_k with $k = 1, 2, 3$ are the principal moments of inertia of the rigid body and the phase variables are the momenta G and L, and their conjugate angular coordinates are g and l.
G represents the magnitude of the angular momentum, while L is the projection of the angular momentum itself onto the principal axis of inertia number 3 of the rigid body. g and l are angles measured in the planes orthogonal respectively to the angular momentum and the third principal axis of inertia.
This being stated, consider the "quasi-gyroscopic" case, in which the principal moment of inertia I_2 is approximately equal to the moment I_1: $I_2 = I_1(1 - \varepsilon)$ with $|\varepsilon| \ll 1$. Under this hypothesis, calculate perturbatively, to the first order in ε, the variations $\Delta\omega_g$ and $\Delta\omega_l$ of the angular velocity components, with respect to the perfect gyroscopic case.

P.49. A satellite moving in the Keplerian central field is subjected to a dissipative force $F = -\alpha \boldsymbol{v}$ with $\alpha \ll 1$. Write the Lagrangian equations of motion and determine the variation of energy and angular momentum over one revolution period.

P.50. Given the Hamiltonian

$$H = \sum \frac{p_k^2}{2} + \omega_k^2 \frac{x_k^2}{2} + \epsilon \sum x_k^4$$

apply the first-order perturbation theory and determine the time variation of the actions, indicating the value of their period.

P.51. Consider a point mass of unit mass subject to the potential

$$V(x) = \begin{cases} \dfrac{1}{2x^2} - \dfrac{1}{x}, & x \geq 1 \\ -\dfrac{1}{2} + \dfrac{(x-1)^2}{2}, & x \leq 1 \end{cases}$$

1. Draw the phase space;
2. determine the period of bounded orbits as a function of energy (hint: use the integral

$$\int \frac{\mathrm{d}x}{\sqrt{2\left(E - \frac{1}{2x^2} + \frac{1}{x}\right)}} = -\frac{1}{2|E|}\sqrt{2x - 1 - 2|E|x^2} + \frac{1}{(2|E|)^{\frac{3}{2}}} \arcsin\left(\frac{2|E|x - 1}{\sqrt{1 - 2|E|}}\right)$$

with $E < 0$);

3. (optional) calculate in implicit form the equation of motion for the bounded orbits.

P.52. A point mass moves on the surface of a cone with a vertical axis and aperture α. Write the Lagrangian of the system, the equations of motion, and determine the first integrals of the system. Determine the constraint reaction in the case of circular orbits.

P.53. Determine the function $f(q)$ such that the transformation $Q = e^p f(q)$, $P = e^{-p} f(q)$ is canonical and calculate its generating function.

P.54. Given the system of two harmonic oscillators described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}(x_1^2 + x_2^2 + 2\alpha x_1 x_2)$$

find the canonical transformation that leads to action-angle variables.

P.55. Solve perturbatively, stopping at the first order, the mechanical problem described by the following Hamiltonian

$$H(A, \varphi) = \omega_0 A + \lambda A^2 \sin^2(\varphi)$$

where A, φ are action-angle variables and λ is the expansion parameter for the perturbative series ($\lambda A/\omega_0 \ll 1$).

P.56. Having defined the Lie operator D_g associated with a function $g(p, q)$ on the phase space as

$$D_g[f(p, q)] \equiv \{f, g\}$$

calculate $\exp(D_g[q])$ and $\exp(D_g[p])$ when $g(p, q) = p$ and when $g(p, q) = q$. Interpret the result as a transformation of the phase space into itself.

P.57. Given the Hamiltonian $H = H_0 + \varepsilon H_1$ with

$$H_0 = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2 + xy}{2}$$

$$H_1 = x^6 + 2y^6$$

perform the canonical transformation that makes H_0 a function of the action variables only and calculate, to the first order in ε, the eigenfrequencies of the system.

P.58. Determine the domain and codomain of the following transformation $T : (p, q) \to (P, Q)$

$$P = P_0\left(e^{\alpha p(1+\beta q)} - 1\right)$$

$$Q = Q_0 \log(1 + \beta q) e^{-\alpha p(1+\beta q)}$$

where P_0, Q_0, α, β are parameters.
Set the conditions for the canonicity of T and determine the generating function $F_2(q, P)$.

P.59. Consider the Hamiltonian

$$H(j,\phi) = \mathrm{J}(1+\varepsilon\phi)$$

(a) Indicate whether the manifold on which the dynamical system described by H is defined is the cylinder $C = \mathbb{R}^1 \times \mathbb{T}^1$, justifying the answer.

(b) Solve exactly Hamilton's equations for arbitrary initial conditions $j(0)$ and $\phi(0)$.

(c) Determine the canonical transformation that reduces H to depend only on J, finding its generating function $F(\mathrm{J},\phi)$.

P.60. Given the coordinate transformation in $\mathbb{R}^n$

$$Q = Aq$$

determine the corresponding canonical transformation in phase space and write its generating function.

P.61. Consider the Hamiltonian

$$H(j,\Phi,t) = \omega j + \epsilon V(j,\Phi,t)$$

where ω is irrational, V is a periodic function in Φ and in time with period 2π.
Determine a time-dependent canonical transformation that reduces H to normal form.
Explicitly calculate the Hamiltonian in normal form and the canonical transformation, at the first perturbative order, in the case where

$$V = j(\sin(\Phi)\cos(t))^2$$

P.62. Explicitly calculate the first eight terms of the canonical transformation defined by the Lie series with generator $G(p,q) = p^2q^2$ and give the explicit expression of the transformation: $P \equiv P(o,q)$ and $Q \equiv Q(p,q)$.

P.63. Apply canonical perturbation theory to the Hamiltonian

$$H = \left[\frac{p_1^2+\omega_1^2x_1^2}{2}\right]^n + \left[\frac{p_2^2+\omega_2^2x_2^2}{2}\right]^n + \frac{\lambda}{2^{2m}}x_1^m x_2^m p_1^m p_2^m$$

stopping at the first order in λ.

P.64. Consider the Hamiltonian

$$H = \sum_{s\geq 1} t^{s-1}\chi_s(p,q)$$

Determine, using the Lie operator associated with H, the time evolution of any dynamical variable A, given its value at an initial time $t=0$.

P.65. Consider the Hamiltonian expressed, in action-angle variables in $\mathbb{R}^2 \times \mathbb{T}^2$, in the form

$$H = H_0(\boldsymbol{j}) + \varepsilon j_1^2 j_2^2 \cos^2(\phi_1)\cos^2(\phi_2)$$

with $H_0(\boldsymbol{j}) = \boldsymbol{\omega}\cdot\boldsymbol{j}$; using canonical perturbation theory to the first order in ε, identify, as a function of the same perturbative parameter, the torus of $H_0(\boldsymbol{j})$ that transforms into a fixed point of the flow generated by H.